Insights into Large Complex Systems via Random Matrix Theory

Lu Wei

SEAS, Harvard

06/23/2016 @ UTC Institute for Advanced Systems Engineering University of Connecticut
Random Matrix Theory
Random Matrix Theory

- milestones
  - Wishart distribution (Wishart [1928])
  - Semicircle law (Wigner [1955])
  - Marchenko-Pastur law (Marchenko-Pastur [1967])
  - Tracy-Widom law (Tracy-Widom [1990s])
  - KPZ universality class (Johansson [2000s])

Tools from almost all branches of mathematics and physics.


Applications: biology, data sciences, economics, information theory, machine learning, wireless communications, ...
Random Matrix Theory

- **milestones**
  - Wishart distribution ([Wishart [1928]](#))

- Other tools from almost all branches of mathematics and physics.
Random Matrix Theory

- milestones
  - Wishart distribution ([Wishart 1928])
  - Semicircle law ([Wigner 1955])
Random Matrix Theory

- **milestones**
  - Wishart distribution (*Wishart [1928]*)
  - Semicircle law (*Wigner [1955]*)
  - Marchenko-Pastur law (*Marchenko-Pastur [1967]*)
Random Matrix Theory

- milestones
  - Wishart distribution (Wishart [1928])
  - Semicircle law (Wigner [1955])
  - Marchenko-Pastur law (Marchenko-Pastur [1967])
  - Tracy-Widom law (Tracy-Widom [1990s])
Random Matrix Theory

- milestones
  - Wishart distribution (Wishart [1928])
  - Semicircle law (Wigner [1955])
  - Marchenko-Pastur law (Marchenko-Pastur [1967])
  - Tracy-Widom law (Tracy-Widom [1990s])
  - KPZ universality class (Johansson [2000s])
Random Matrix Theory

- milestones
  - Wishart distribution (Wishart [1928])
  - Semicircle law (Wigner [1955])
  - Marchenko-Pastur law (Marchenko-Pastur [1967])
  - Tracy-Widom law (Tracy-Widom [1990s])
  - KPZ universality class (Johansson [2000s])

- tools from almost all branches of mathematics and physics
Random Matrix Theory

**milestones**
- Wishart distribution (*Wishart [1928]*)
- Semicircle law (*Wigner [1955]*)
- Marchenko-Pastur law (*Marchenko-Pastur [1967]*)
- Tracy-Widom law (*Tracy-Widom [1990s]*)
- KPZ universality class (*Johansson [2000s]*)

**tools from almost all branches of mathematics and physics**
- *Akemann et al. (eds) [2011]* The Oxford Handbook of Random Matrix Theory. *Oxford University Press*
Random Matrix Theory

- **milestones**
  - Wishart distribution (Wishart [1928])
  - Semicircle law (Wigner [1955])
  - Marchenko-Pastur law (Marchenko-Pastur [1967])
  - Tracy-Widom law (Tracy-Widom [1990s])
  - KPZ universality class (Johansson [2000s])

- **tools from almost all branches of mathematics and physics**

- **applications**: biology, data sciences, economics, information theory, machine learning, wireless communications, . . .
Product of Random Matrices

joint works with Akemann, Hero, Kieburg, Liu, Tarokh, Zhang, Zheng
Products of Random Matrices

$y = Hx + w$, where $H = H_n \cdots H_2 H_1$

$\text{Tx. Rx. cluster 1 cluster 2 cluster n-1}$

$w$
Products of Random Matrices

- Crisanti et al. [1993] Products of Random Matrices in Statistical Physics. *Springer*
Products of Random Matrices

- Crisanti et al. [1993] Products of Random Matrices in Statistical Physics. *Springer*
Products of Random Matrices

- Crisanti et al. [1993] Products of Random Matrices in Statistical Physics. *Springer*

\[ y = Hx + w, \text{ where } H = H_n \cdots H_2 H_1 \]
**Products of Random Matrices**

- **Crisanti et al. [1993]** Products of Random Matrices in Statistical Physics. *Springer*

\[ y = Hx + w, \text{ where } H = H_n \cdots H_2 H_1 \]

Diagram:

- **Tx.** → **cluster 1** → **H_1** → **cluster 2** → **H_2** → **cluster n-1** → **H_n** → **Rx.**
- **w** → **y**
Products of Random Matrices

- **Crisanti et al. [1993]** Products of Random Matrices in Statistical Physics. *Springer*


\[ y = Hx + w, \text{ where } H = H_n \cdots H_2 H_1 \]

an earlier attempt via eigenvalues and singular values relation*

---

Products of Random Matrices

\[ H = H_n \cdots H_2 H_1 \]
Products of Random Matrices

\[ H = H_n \cdots H_2 H_1 \]

- \( n = 1 \) – Wishart-Laguerre ensemble Bronk [1965]
Products of Random Matrices

\[ H = H_n \cdots H_2 H_1 \]

- \( n = 1 \) – Wishart-Laguerre ensemble Bronk [1965]

\[ \rho (\lambda_1, \ldots, \lambda_m) \propto \det^2 \left( \lambda_k^{i-1} \right) \prod_{i=1}^{m} e^{-\lambda_i} \]
Products of Random Matrices

\[ H = H_n \cdots H_2 H_1 \]

- \( n = 1 \) — Wishart-Laguerre ensemble Bronk [1965]
  \[ p(\lambda_1, \ldots, \lambda_m) \propto \det^2 \left( \lambda_{k}^{-1} \right) \prod_{i=1}^{m} e^{-\lambda_i} \]

- \( n \) arbitrary\(^\dagger\)

Products of Random Matrices

\[ \mathbf{H} = \mathbf{H}_n \cdots \mathbf{H}_2 \mathbf{H}_1 \]

- \( n = 1 \) – Wishart-Laguerre ensemble Bronk [1965]
  \[ p (\lambda_1, \ldots, \lambda_m) \propto \det^2 \left( \lambda_k^{j-1} \right) \prod_{i=1}^{m} e^{-\lambda_i} \]

- \( n \) arbitrary\(^\dagger\)
  \[ p (\lambda_1, \ldots, \lambda_m) \propto \det \left( \lambda_k^{j-1} \right) \det \left( f_j(\lambda_k) \right) \]

Products of Random Matrices

\[ H = H_n \cdots H_2 H_1 \]

- \( n = 1 \) – Wishart-Laguerre ensemble Bronk [1965]
  \[ p(\lambda_1, \ldots, \lambda_m) \propto \det^2 \left( \lambda_k^{j-1} \right) \prod_{i=1}^{m} e^{-\lambda_i} \]

- \( n \) arbitrary

  \[ p(\lambda_1, \ldots, \lambda_m) \propto \det \left( \lambda_k^{j-1} \right) \det (f_j(\lambda_k)) \]

  \[ f_j(x) = G_{0,m}^{m,0} \left( \begin{array}{c|c} x & \vdots \\ \hline 0, \ldots, 0, j-1 \end{array} \right) = \frac{1}{2\pi i} \oint_{\mathcal{L}} \frac{d u}{u} x^{-u} \Gamma^{m-1}(u) \Gamma(u+j-1) \]

\[ \text{\textsuperscript{†} Akemann, Kieburg, W. [2013] Singular value correlation functions for products of Wishart random matrices, J. Phys. A} \]
Applications to Capacity Analysis

\[ \sum_{i=1}^{m} \log \left( 1 + \gamma \lambda_i \right) \]

outage capacity of orthogonal space-time codes, i.e., distribution of

\[ \sum_{i=1}^{m} \lambda_i \]

outage capacity of double-cluster channels, i.e., distribution of

\[ \sum_{i=1}^{m} \log \left( 1 + \gamma \lambda_i \right) \]
Applications to Capacity Analysis

- ergodic capacity Akemann-Kieburg-W. [2013]

\[
\mathbb{E} \left[ \sum_{i=1}^{m} \log (1 + \gamma \lambda_i) \right]
\]
Applications to Capacity Analysis

- ergodic capacity Akemann-Kieburg-W. [2013]
  \[
  \mathbb{E} \left[ \sum_{i=1}^{m} \log (1 + \gamma \lambda_i) \right]
  \]

- outage capacity of orthogonal space-time codes\footnote{W., Zheng, Corander, Taricco [2015] On the outage capacity of orthogonal space-time block codes over multi-cluster scattering MIMO channels, IEEE Trans. Commun.}, i.e., distribution of
  \[
  \sum_{i=1}^{m} \lambda_i
  \]
Applications to Capacity Analysis

- ergodic capacity Akemann-Kieburg-W. [2013]

\[
\mathbb{E} \left[ \sum_{i=1}^{m} \log (1 + \gamma \lambda_i) \right]
\]

- outage capacity of orthogonal space-time codes\(^\dagger\), i.e., distribution of

\[
\sum_{i=1}^{m} \lambda_i
\]

- outage capacity of double-cluster channels\(^\S\), i.e., distribution of

\[
\sum_{i=1}^{m} \log (1 + \gamma \lambda_i) \quad \text{for } n = 2
\]


Spiked Products of Random Matrices: Phase Transitions
Spiked Products of Random Matrices: Phase Transitions

\[ H = \Sigma^{1/2}H_n \cdots H_2H_1 \]
Spiked Products of Random Matrices: Phase Transitions

\[ H = \Sigma^{1/2} H_n \cdots H_2 H_1 \]

- natural generalization \( \mathbb{E} \left[ HH^\dagger \right] \propto \Sigma = \text{diag} (\sigma_1, \ldots, \sigma_m) \) – spikes
Spiked Products of Random Matrices: Phase Transitions

\[ H = \sum^{1/2} H_n \cdots H_2 H_1 \]

- natural generalization \( \mathbb{E} \left[ HH^\dagger \right] \propto \Sigma = \text{diag} (\sigma_1, \ldots, \sigma_m) \) – spikes
Spiked Products of Random Matrices: Phase Transitions

\[ H = \Sigma^{1/2} H_n \cdots H_2 H_1 \]

- natural generalization \( \mathbb{E} \left[ HH^\dagger \right] \propto \Sigma = \text{diag} (\sigma_1, \ldots, \sigma_m) - \text{spikes} \)
- \( n = 1 - \text{Baik-Ben Arous-Péché [2005]} \) Phase transition of the largest eigenvalue for nonnull complex sample covariance matrices, Ann. Probab.
  - \( \lambda_1 \) from Tracy-Widom to Gaussian (BBP phase transition)
Spiked Products of Random Matrices: Phase Transitions

\[ H = \Sigma^{1/2} H_n \cdots H_2 H_1 \]

- natural generalization \( \mathbb{E} [HH^\dagger] \propto \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_m) \) – spikes
  - \( \lambda_1 \) from Tracy-Widom to Gaussian (BBP phase transition)
  - critical value \( \sigma_{\text{crit}} = 2 \)
Spiked Products of Random Matrices: Phase Transitions

\[ H = \sum^{1/2} H_n \cdot \cdot \cdot H_2 H_1 \]

- natural generalization \( \mathbb{E} \left[ HH^\dagger \right] \propto \Sigma = \text{diag} (\sigma_1, \ldots, \sigma_m) \) – spikes

  - \( \lambda_1 \) from Tracy-Widom to Gaussian (BBP phase transition)
  - critical value \( \sigma_{\text{crit}} = 2 \)

- \( n \) arbitrary

---

\( \dagger \) Liu, W., Zhang Singular values for spiked products of complex Ginibre matrices
Spiked Products of Random Matrices: Phase Transitions

\[ H = \Sigma^{1/2} H_n \cdots H_2 H_1 \]

- natural generalization \( \mathbb{E} \left[ HH^\dagger \right] \propto \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_m) \) – spikes
  - \( \lambda_1 \) from Tracy-Widom to Gaussian (BBP phase transition)
  - critical value \( \sigma_{\text{crit}} = 2 \)
- \( n \) arbitrary\(^\dagger\)
  - \( \lambda_1 \) BBP phase transition

\(^\dagger\)Liu, W., Zhang Singular values for spiked products of complex Ginibre matrices

---

Lu Wei

Random Matrix Theory and Large Complex Systems
Spiked Products of Random Matrices: Phase Transitions

\[ H = \sum^{1/2} H_n \cdots H_2 H_1 \]

- natural generalization \( \mathbb{E} \left[ HH^\dagger \right] \propto \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_m) \) – spikes
  - \( \lambda_1 \) from Tracy-Widom to Gaussian (BBP phase transition)
  - critical value \( \sigma_{\text{crit}} = 2 \)
- \( n \) arbitrary
  - \( \lambda_1 \) BBP phase transition
  - critical value \( \sigma_{\text{crit}} = n + 1 \)

\[ \text{Liu, W., Zhang Singular values for spiked products of complex Ginibre matrices} \]
Phase Transitions
Phase Transitions

- MIMO radar detection

\[ \parallel W, Zheng, Hero, Tarokh \parallel \text{ Scaling laws and phase transitions for target detection in MIMO radar, ITW'16} \]
Phase Transitions

- MIMO radar detection; community detection, …

---

‖ W., Zheng, Hero, Tarokh Scaling laws and phase transitions for target detection in MIMO radar, ITW’16
Applications to Other Large Complex Systems
Signal Processing

joint works with Dharmawansa, Liang, McKay, Tirkkonen
Signal Detection

- **cognitive radio** - a solution to spectrum underutilization problem by dynamic spectrum access

\[
Y_{m \times N} = \begin{bmatrix} y_1, y_2, \ldots, y_N \end{bmatrix} \Rightarrow \ \{ H_0 : \text{noise} \ H_1 : \text{signal} + \text{noise} \}
\]

- Data sample covariance: \( R = YY^\dagger \)
- Noise sample covariance: \( E \)

Lu Wei
Random Matrix Theory and Large Complex Systems 12 / 19
Signal Detection

- **cognitive radio** - a solution to spectrum underutilization problem by dynamic spectrum access
  - unlicensed users are allowed to opportunistically use the frequency bands that are not heavily occupied by licensed users
Signal Detection

- **cognitive radio** - a solution to spectrum underutilization problem by dynamic spectrum access
  - unlicensed users are allowed to opportunistically use the frequency bands that are not heavily occupied by licensed users
- awareness of spectrum usage information via **spectrum sensing**

\[
Y_{m \times 1} = H_{m \times p} x + w
\]

\[
Y_{m \times N} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} \Rightarrow \begin{cases} H_0: \text{noise} \\ H_1: \text{signal} + \text{noise} \end{cases}
\]

\[
R = YY^\dagger, \text{data sample covariance; } E = \text{noise sample covariance}
\]
Signal Detection

- **cognitive radio** - a solution to spectrum underutilization problem by dynamic spectrum access
  - unlicensed users are allowed to opportunistically use the frequency bands that are not heavily occupied by licensed users
- awareness of spectrum usage information via **spectrum sensing**

\[
\mathbf{y} = \mathbf{H} \mathbf{x} + \mathbf{w}
\]

\(m \times 1\) \( m \times p\)
**Signal Detection**

- **cognitive radio** - a solution to spectrum underutilization problem by dynamic spectrum access
  - unlicensed users are allowed to opportunistically use the frequency bands that are not heavily occupied by licensed users
- awareness of spectrum usage information via **spectrum sensing**

\[
y_{m \times 1} = H_{m \times p} x + w_{m \times p}
\]

\[
Y_{m \times N} = (y_1, y_2, \ldots, y_N) \quad \Rightarrow \quad \begin{cases}
H_0 : \text{noise} \\
H_1 : \text{signal + noise}
\end{cases}
\]
Signal Detection

- **cognitive radio** - a solution to spectrum underutilization problem by dynamic spectrum access
  - unlicensed users are allowed to opportunistically use the frequency bands that are not heavily occupied by licensed users
- awareness of spectrum usage information via **spectrum sensing**
  \[
  \mathbf{y} = \mathbf{H} \mathbf{x} + \mathbf{w},
  \]
  \[
  \mathbf{Y} = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_N) \Rightarrow \begin{cases} \mathcal{H}_0 & : \text{noise} \\ \mathcal{H}_1 & : \text{signal + noise} \end{cases}
  \]
- \( R = \mathbf{Y}\mathbf{Y}^\dagger \) data sample covariance; \( \mathbf{E} \) noise sample covariance
Signal Detection

- $p > 1$ but unknown\(^*\)

$$T_{ST} = \frac{\det(R)}{\left(\frac{1}{m}\text{tr}(R)\right)^m}$$

Signal Detection

- $p > 1$ but unknown$^*$

\[ T_{ST} = \frac{\text{det}(\mathbf{R})}{\left(\frac{1}{m}\text{tr}(\mathbf{R})\right)^m} \]

- low signal-to-noise ratio$^\dagger$

\[ T_J = \frac{\text{tr}(\mathbf{R}^2)}{\text{tr}^2(\mathbf{R})} \]


Signal Detection

- $p > 1$ but unknown
  
  $$T_{ST} = \frac{\det(R)}{\left(\frac{1}{m}\text{tr}(R)\right)^m}$$

- low signal-to-noise ratio
  
  $$T_J = \frac{\text{tr}(R^2)}{\text{tr}^2(R)}$$

- unknown noise covariance
  
  $$T_W = \frac{\det(E)}{\det(R + E)}$$

---


Coding Theory

joint works with Corander, Pitaval, Tirkkonen
fundamental issue: cardinality and minimum distance tradeoff
fundamental issue: cardinality and minimum distance tradeoff

a code with cardinality $|C|$, $C = \{G_1, G_2, \ldots, G_{|C|}\} \subset G$
fundamental issue: cardinality and minimum distance tradeoff

a code with cardinality $|C|$, $C = \{G_1, G_2, \ldots, G_{|C|}\} \subset G$

minimum distance, $r = \min \left\{ \|G_i - G_j\| \mid G_i, G_j \in C, i \neq j \right\}$
fundamental issue: cardinality and minimum distance tradeoff

a code with cardinality $|C|$, $C = \{G_1, G_2, \ldots, G_{|C|}\} \subset \mathcal{G}$

minimum distance, $r = \min \left\{ \|G_i - G_j\| \mid G_i, G_j \in C, i \neq j \right\}$

$$\frac{1}{\mu(B(r))} \leq |C| \leq \frac{1}{\mu(B(r/2))}$$

Gilbert-Varshamov bound

Hamming bound
**fundamental issue**: cardinality and minimum distance tradeoff

a code with cardinality $|C|$, $C = \{G_1, G_2, \ldots, G_{|C|}\} \subset G$

minimum distance, $r = \min \left\{ \|G_i - G_j\| \mid G_i, G_j \in C, i \neq j \right\}$

$$\frac{1}{\mu(B(r))} \leq |C| \leq \frac{1}{\mu(B(r/2))}$$

Gilbert-Varshamov bound

Hamming bound

metric ball, $B(r) = \{G \in G \mid d(G, G') \leq r\}, G' \in G$
fundamental issue: cardinality and minimum distance tradeoff

a code with cardinality $|C|$, $C = \{ \mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_{|C|} \} \subset \mathcal{G}$

minimum distance, $r = \min \left\{ ||\mathbf{G}_i - \mathbf{G}_j|| \mid \mathbf{G}_i, \mathbf{G}_j \in C, i \neq j \right\}$

$$\frac{1}{\mu(B(r))} \leq |C| \leq \frac{1}{\mu(B(r/2))}$$

Gilbert-Varshamov bound

Hamming bound

metric ball, $B(r) = \{ \mathbf{G} \in \mathcal{G} \mid d(\mathbf{G}, \mathbf{G}') \leq r \}, \mathbf{G}' \in \mathcal{G}$

volume of metric ball, $\mu(B(r)) = \int_{d(\mathbf{G}, \mathbf{G}') \leq r} f(\mathbf{G}) \ d\mathbf{G}$
Volume of Metric Balls
Unitary Group

\[ \mu(B(r)) \propto \int \cdots \int ||U - I||_{F \leq r} \prod_{1 \leq j < k \leq n} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{i=1}^n d\theta_i \]


W., Pitaval, Corander, Tirkkonen From random matrix theory to coding theory: Volume of a metric ball in unitary group, IEEE Trans. Inf. Theory, submitted, arXiv:1506.07259 as \( n \to \infty \) 

\[ \mu(B(r)) \approx \frac{1}{2} \text{erf}(n) - \frac{1}{2} \text{erf}(n - \frac{r}{2}) \]

CLT of linear statistics of unitary group

Lu Wei

Random Matrix Theory and Large Complex Systems 17 / 19
Unitary Group

\[ \mu(B(r)) \propto \int \cdots \int \prod_{1 \leq j < k \leq n} \left| e^{i\theta_j} - e^{i\theta_k} \right|^2 \prod_{i=1}^{n} d\theta_i \]
Unitary Group

\[ \mu(B(r)) \propto \int \cdots \int_{\|U-I_n\|_F \leq r} \prod_{1 \leq j < k \leq n} \left| e^{i \theta_j} - e^{i \theta_k} \right|^2 \prod_{i=1}^{n} d\theta_i \]

Unitary Group

\[ \mu(B(r)) \propto \int \cdots \int_{\|U-I_n\|_F \leq r} \prod_{1 \leq j < k \leq n} \left| e^{i\theta_j} - e^{i\theta_k} \right|^2 \prod_{i=1}^{n} d\theta_i \]


- limiting behavior* as \( n \to \infty \)

\[ \mu(B(r)) \approx \frac{1}{2} \text{erf}(n) - \frac{1}{2} \text{erf} \left( n - \frac{r^2}{2} \right) \]

Unitary Group

\[ \mu(B(r)) \propto \int \cdots \int_{\|U-I_n\|_F \leq r} \prod_{1 \leq j < k \leq n} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{i=1}^{n} d\theta_i \]


- Limiting behavior* as \( n \to \infty \)

\[ \mu(B(r)) \approx \frac{1}{2} \text{erf}(n) - \frac{1}{2} \text{erf} \left( n - \frac{r^2}{2} \right) \]

- Super-exponential rate of convergence \( O(n^{-cn}) \)

Unitary Group

\[ \mu(B(r)) \propto \int \cdots \int_{\|u-I_n\|_F \leq r} \prod_{1 \leq j < k \leq n} \left| e^{i\theta_j} - e^{i\theta_k} \right|^2 \prod_{i=1}^n d\theta_i \]


- limiting behavior* as \( n \to \infty \)

\[ \mu(B(r)) \simeq \frac{1}{2} \text{erf}(n) - \frac{1}{2} \text{erf} \left( n - \frac{r^2}{2} \right) \]

- super-exponential rate of convergence \( O(n^{-cn}) \)
- CLT of linear statistics of unitary group

Grassmann Manifold
Grassmann Manifold

\[ \mu(B(r)) \propto \int \cdots \int_{0 \leq x_i \leq 1} \prod_{1 \leq j < k \leq p} (x_j - x_k)^2 \prod_{i=1}^{p} x_i^{n-p-q} (1 - x_i)^{q-p} \, dx_i \]
Grassmann Manifold

\[ \mu(B(r)) \propto \int \cdots \int_{0 \leq x_i \leq 1} \prod_{1 \leq j < k \leq p} (x_j - x_k)^2 \prod_{i=1}^{p} x_i^{n-p-q} (1 - x_i)^{q-p} \, dx_i \]

- Barg-Nogin [2002] fixed \( p, q \), large \( n \); Dai et al. [2008] \( r \leq 1 \)
Grassmann Manifold

\[ \mu(B(r)) \propto \int \cdots \int_{0 \leq x_i \leq 1} \prod_{1 \leq j < k \leq p} (x_j - x_k)^2 \prod_{i=1}^{p} x_i^{n-p-q} (1 - x_i)^{q-p} \, dx_i \]

- Barg-Nogin [2002] fixed \( p, q, \) large \( n \); Dai et al. [2008] \( r \leq 1 \)

- Limiting behavior\(^\dagger\) as \( n, p, q \to \infty \)

\[ \mu(B(r)) \approx \frac{1}{2} \text{erf} \left( \frac{b}{\sqrt{2a}} \right) - \frac{1}{2} \text{erf} \left( \frac{b - r^2}{\sqrt{2a}} \right) \]

Grassmann Manifold

\[ \mu(B(r)) \propto \int \cdots \int_{0 \leq x_i \leq 1} \prod_{1 \leq j < k \leq p} (x_j - x_k)^2 \prod_{i=1}^{p} x_i^{n-p-q} (1 - x_i)^{q-p} \, dx_i \]

- Barg-Nogin [2002] fixed \( p, q \), large \( n \); Dai et al. [2008] \( r \leq 1 \)

- Limiting behavior\(^\dagger\) as \( n, p, q \to \infty \)

\[ \mu(B(r)) \simeq \frac{1}{2} \text{erf} \left( \frac{b}{\sqrt{2a}} \right) - \frac{1}{2} \text{erf} \left( \frac{b - r^2}{\sqrt{2a}} \right) \]

- CLT of linear statistics of Jacobi ensemble

---

Grassmann Manifold

\[ \mu(B(r)) \propto \int \cdots \int_{0 \leq x_i \leq 1} \prod_{0 \leq x_i \leq 1} (x_j - x_k)^2 \prod_{1 \leq j < k \leq p} x_i^{n-p-q} (1 - x_i)^{q-p} \, dx_i \]

- Barg-Nogin [2002] fixed \( p, q, \) large \( n; \) Dai et al. [2008] \( r \leq 1 \)
- Limiting behavior\( ^{\dagger} \) as \( n, p, q \to \infty \)

\[ \mu(B(r)) \simeq \frac{1}{2} \text{erf}\left(\frac{b}{\sqrt{2a}}\right) - \frac{1}{2} \text{erf}\left(\frac{b - r^2}{\sqrt{2a}}\right) \]

- CLT of linear statistics of Jacobi ensemble
- Moments from Painlevé V

Random Matrix Theory and Large Complex Systems

Lu Wei

Random Matrix Theory and Large Complex Systems 19 / 19